Warsaw University of Technology

Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

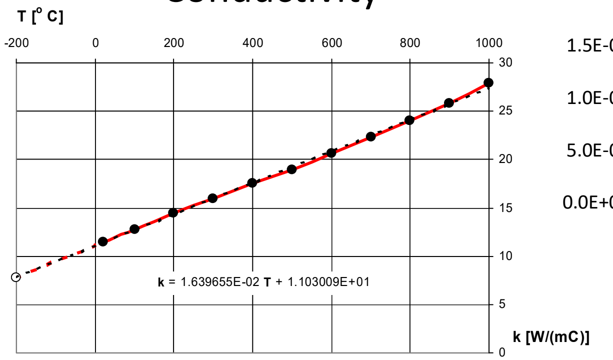
Heat transfer and thermal stresses

10.2021

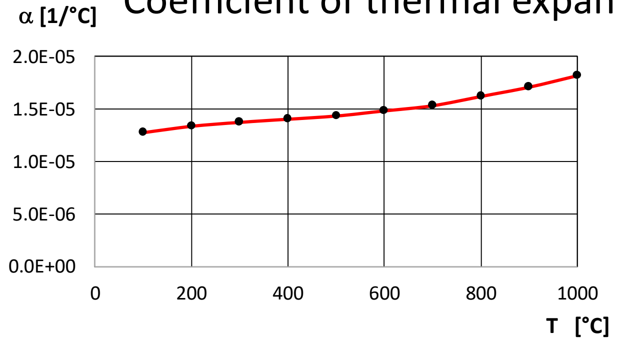
Thermo-mechanical properties versus temperature

Nimonic 90: nickel-based high-temperature low creep alloy for use in aircraft and gas turbine components such as turbine blades and engine exhaust jets.

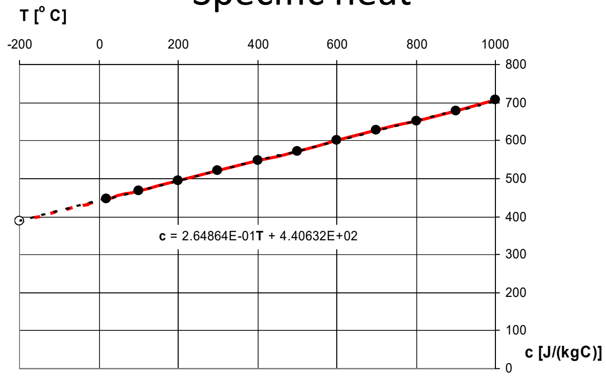
Conductivity



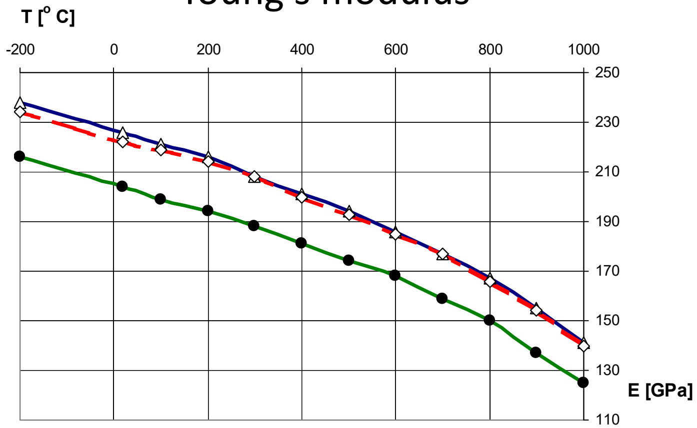
Coefficient of thermal expansion



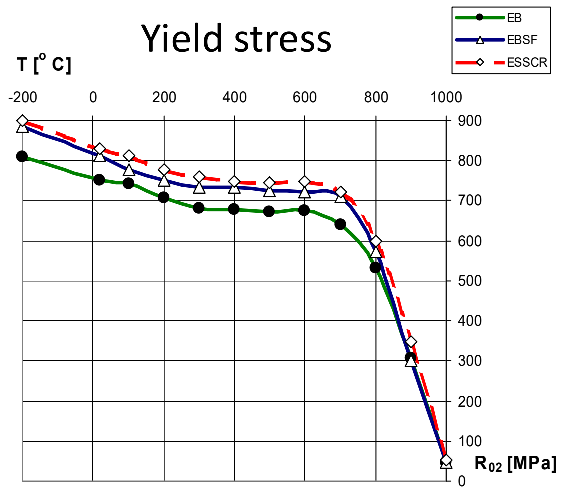
Specific heat



Young's modulus



Yield stress



EB- extruded bar
EBSC - extruded bar subsequently forged
ESSCR – extruded section subsequently cold rolled

www.specialmetals.com

Secant coefficient of thermal expansion $\alpha_{se}(T)$

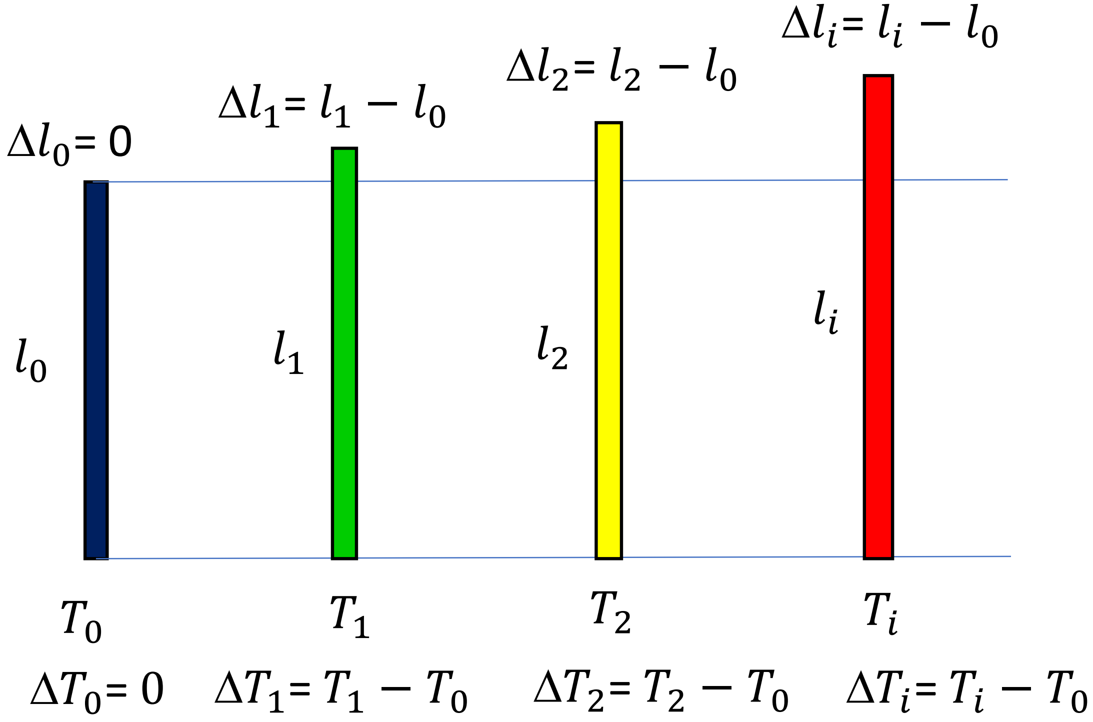
Thermal strain: $\varepsilon_T = \frac{\Delta l}{l} = \alpha_{se} \cdot \Delta T$

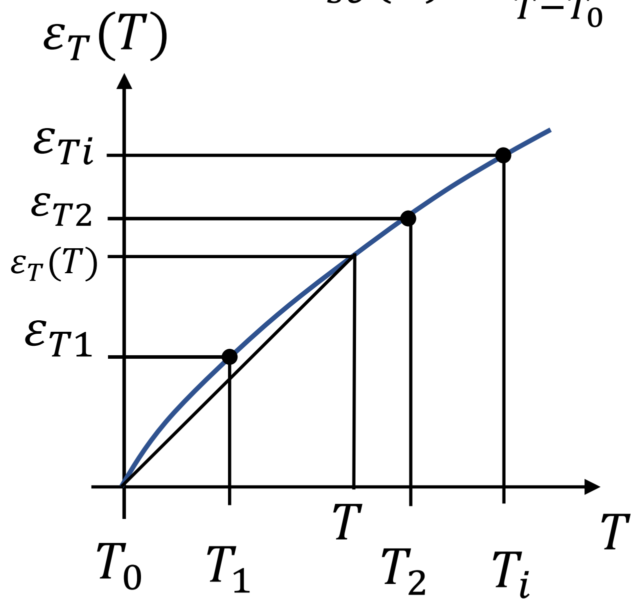
↖ secant coefficient of thermal expansion

$$\varepsilon_{Ti} = \frac{\Delta l_i}{l_0}$$

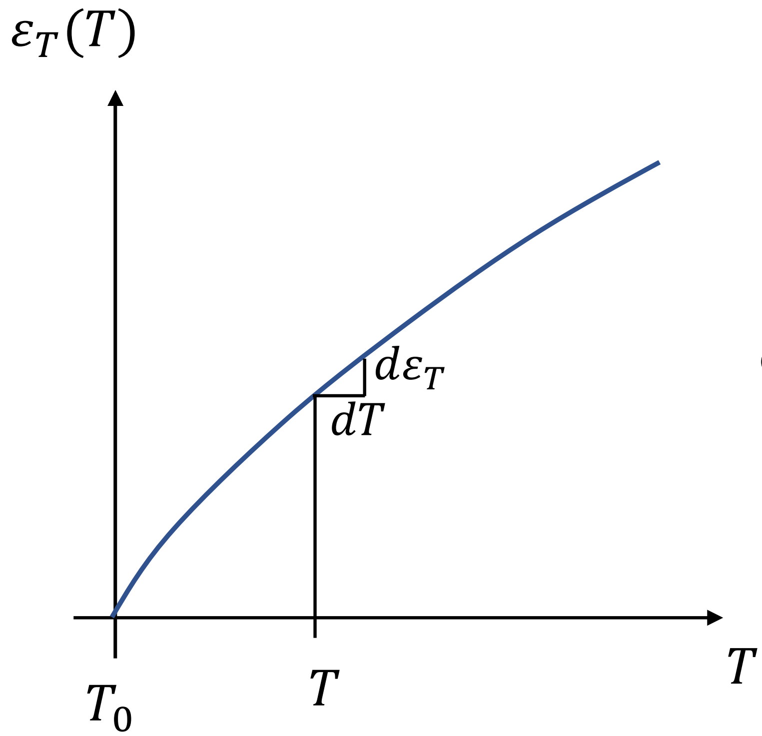
$$\alpha_{se}(T_i) = \frac{\varepsilon_{Ti}}{\Delta T_i} = \frac{\Delta l_i}{l_0 \Delta T_i}$$

$$\alpha_{se}(T) = \frac{\varepsilon_T(T)}{T - T_0}$$





Instantinous coefficient of thermal expansion $\alpha_{ins}(T)$



$$\alpha_{ins}(T) = \frac{d\varepsilon_T}{dT}$$

Coefficient of thermal expansion

T_0 - temperature at which $\varepsilon_T = 0$ - in the test

T_{REF} - temperature at which $\varepsilon_T = 0$ - for working conditions

for $T_{REF} = T_0$:

$$\alpha_{se}(T) = \frac{1}{(T - T_0)} \int_{T_0}^{T} \alpha_{ins}(\overline{T})\, d\overline{T}$$

$$\varepsilon_T(T) = \alpha_{se}(T) \cdot (T - T_0)$$

if $T_{REF} \neq T_0$, the secant coefficient $\alpha_{se}(T)$ is recalculated

Strain in a structure including the thermal effect

The total strain is a sum of thermal and elastic strain:

$$\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times 1}{\{\varepsilon\}_T} + \underset{6\times 1}{\{\varepsilon\}_e} = \begin{Bmatrix} \alpha_x \Delta T \\ \alpha_y \Delta T \\ \alpha_z \Delta T \\ 0 \\ 0 \\ 0 \end{Bmatrix}_T + \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}_e = \begin{Bmatrix} \alpha_x \Delta T \\ \alpha_y \Delta T \\ \alpha_z \Delta T \\ 0 \\ 0 \\ 0 \end{Bmatrix}_T + \underset{6\times 6}{[D]^{-1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

$$\underset{6\times 1}{\{\varepsilon\}_e} = \underset{6\times 6}{[D]^{-1}} \underset{6\times 1}{\{\sigma\}}$$

inverse constitutive matrix

stress vector

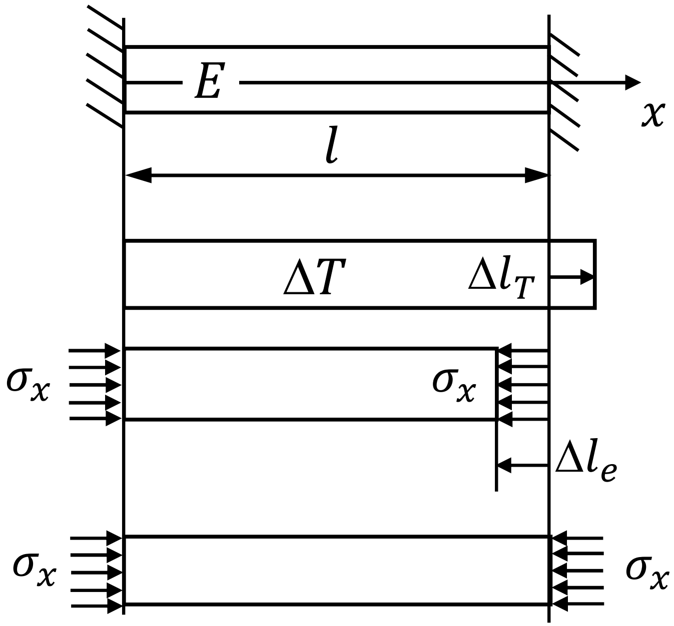
For isotropy:

$$\underset{6\times 1}{\{\varepsilon\}} = \alpha_{se} \cdot \Delta T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_T + \underset{6\times 6}{[D]^{-1}} \cdot \underset{6\times 1}{\{\sigma\}}$$

Thermal stress is observed for:

- nonuniform temperature field
- temperature change and nonhomogeneous material
- temperature change and statically indeterminate constraints

Example: statically indeterminate bar with uniform temperature distribution



$$\Delta l_T + \Delta l_e = 0$$

$$\varepsilon_{xT} \cdot l + \varepsilon_{xe} \cdot l = 0$$

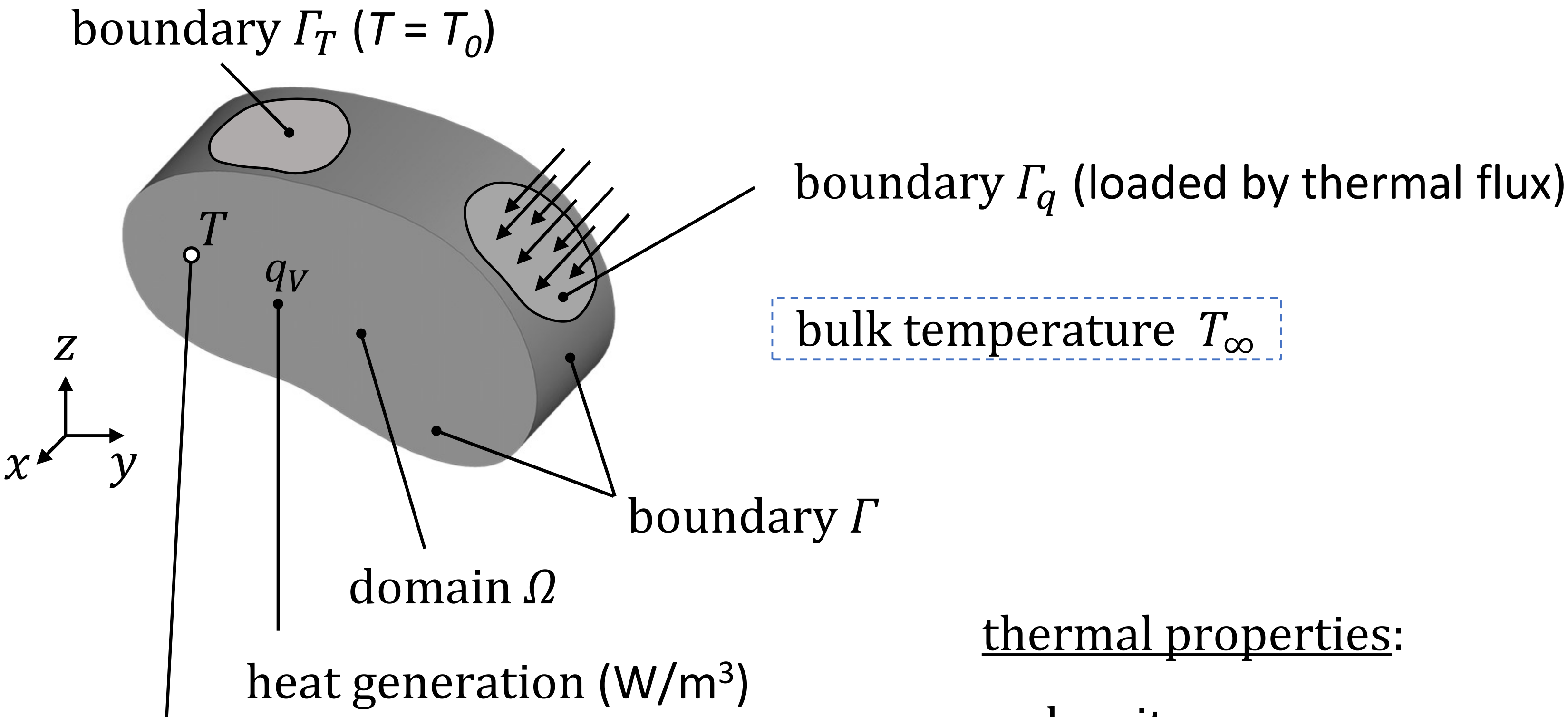
$$\alpha_{se} \cdot \Delta T \cdot l + \frac{\sigma_x}{E} \cdot l = 0$$

$$\sigma_x = -E \cdot \alpha_{se} \cdot \Delta T$$

for: $E = 2 \cdot 10^5$ MPa, $\Delta T = 100°C$, $\alpha_{se} = 1.2 \cdot 10^{-5}$ 1/ °C:

$\sigma_x = -240$ MPa (compression – possible buckling)

Boundary value problem of heat transfer



UNKNOWN FUNCTION
temperature $T(x, y, z, t)$

thermal properties:

- density ρ
- conductivities λ_x , λ_y , λ_z
- specific heat c

Thermal analysis

Transient heat flow inside a solid body (law of conservation of energy):

$$\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + q_V\ (x, y, z, t) = \rho c\ \frac{\partial T}{\partial t}$$

$\rho c\ \frac{\partial T}{\partial t} = 0$ for a steady state:

Fquivalent functional for steady state analysis :

$$J = \int_{\Omega} \frac{1}{2}\left(\lambda_x \left(\frac{\partial T}{\partial x}\right)^2 + \lambda_y \left(\frac{\partial T}{\partial y}\right)^2 + \lambda_z \left(\frac{\partial T}{\partial z}\right)^2 - 2q_V T\right) d\Omega +$$

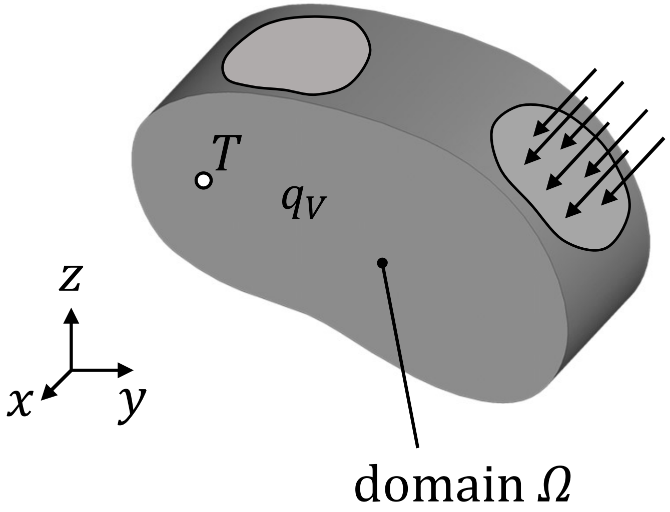
$$+\int_{\Gamma_q} \left(\frac{1}{2}\alpha(T - T_\infty)^2 + qT\right) d\Gamma_q$$

For isotropy:

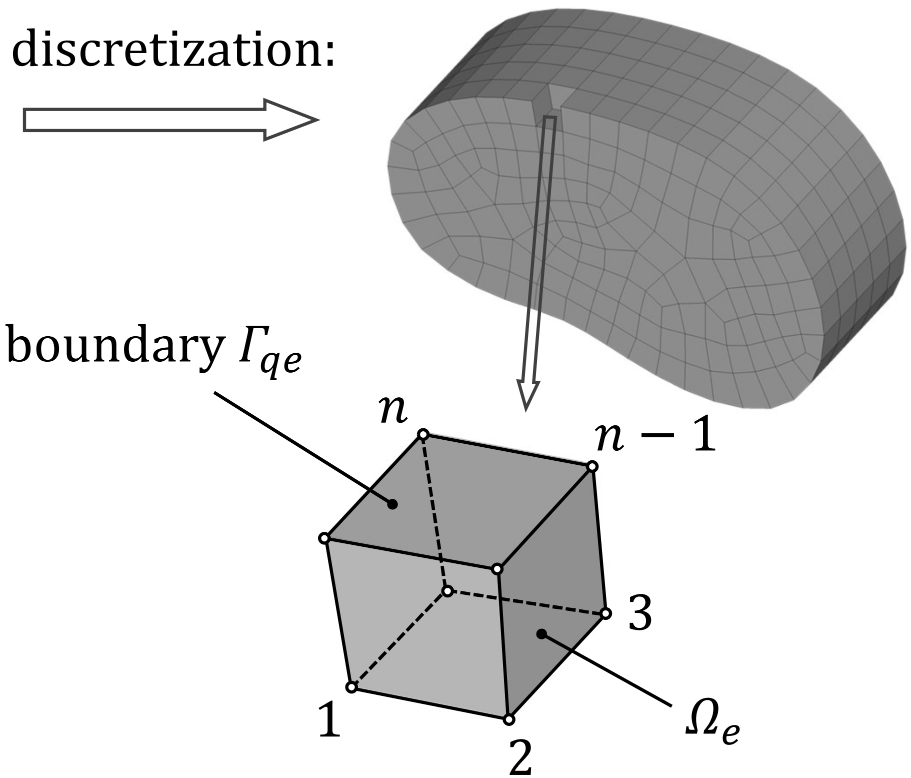
$$J = \int_{\Omega} \frac{1}{2}\left(\lambda\left(\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right) - 2q_V T\right) d\Omega + \int_{\Gamma_q} \left(\frac{1}{2}\alpha(T - T_\infty)^2 + qT\right) d\Gamma_q$$

Finite element model

NOE – no. of finite elements
NON – no. of nodes



discretization:

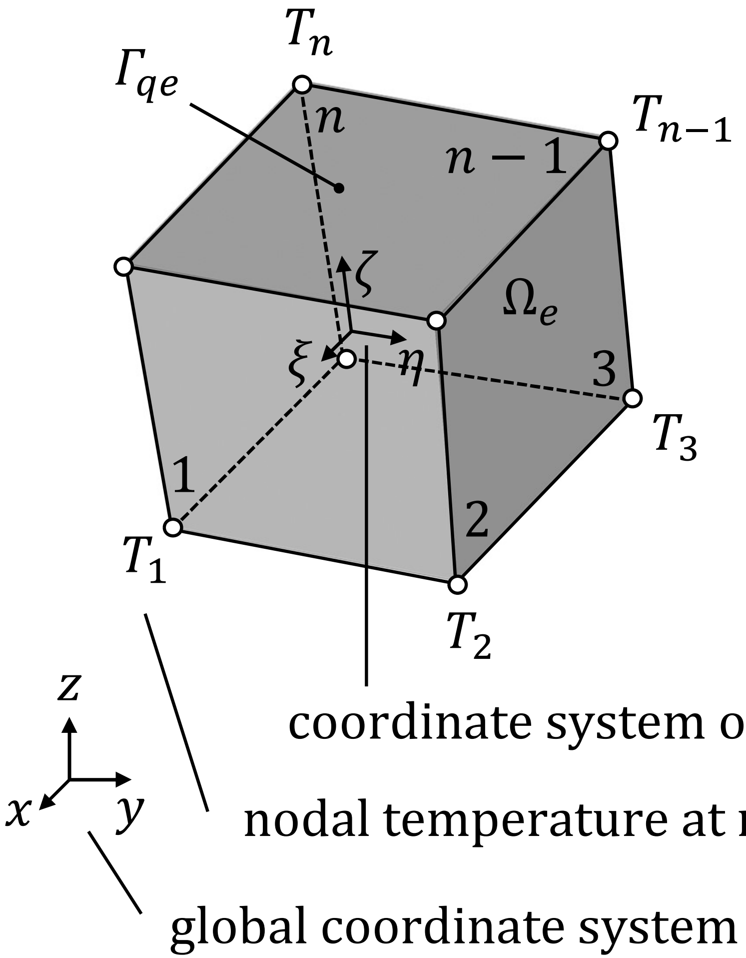


$$\Omega = \sum_{e=1}^{NOE} \Omega_e \text{ and } \Omega_i \underset{i \neq j}{\cap} \Omega_j = 0$$

Finite element with n - nodes

Nodal approximation in the finite element for thermal analysis

$$\text{temperature}\;\; T(\xi,\eta,\zeta) = \underset{1\times n}{\lfloor N(\xi,\eta,\zeta)\rfloor}\underset{n\times 1}{\{T\}_e}$$



Vector of shape functions

$$\underset{1\times n}{\lfloor N(\xi,\eta,\zeta)\rfloor} = \lfloor N_1, N_2, \dots, N_i, \dots, N_n \rfloor$$

Local vector of nodal temperatures

$$\underset{n\times 1}{\{T\}_e} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_i \\ \vdots \\ T_n \end{Bmatrix}_e$$

Functional formulation for a finite element (isotropy and steady state):

$$J_e = \int_{\Omega_e} \frac{1}{2}\left(\lambda\left(\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right) - 2q_V T\right) d\Omega_e + \int_{\Gamma_{qe}} \left(\frac{1}{2}\alpha(T - T_\infty)^2 + qT\right) d\Gamma_{qe}$$

derivative of J_e with respect to temperature T_i:

$$\frac{\partial J_e}{\partial T_i} = \int_{\Omega_e} \left(\lambda\left(\frac{\partial T}{\partial x}\cdot\frac{\partial}{\partial T_i}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial T}{\partial y}\cdot\frac{\partial}{\partial T_i}\left(\frac{\partial T}{\partial y}\right) + \frac{\partial T}{\partial z}\cdot\frac{\partial}{\partial T_i}\left(\frac{\partial T}{\partial z}\right)\right) - q_V \frac{\partial T}{\partial T_i}\right) d\Omega_e + \int_{\Gamma_{qe}} \left(\alpha(T - T_\infty)\frac{\partial T}{\partial T_i} + q\frac{\partial T}{\partial T_i}\right) d\Gamma_{qe}$$

$$T(\xi, \eta, \zeta) = \underset{1\times n}{\lfloor N(\xi, \eta, \zeta)\rfloor}\underset{n\times 1}{\{T\}_e} = \lfloor N_1, N_2, \dots, N_i, \dots, N_n \rfloor \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_i \\ \vdots \\ T_n \end{Bmatrix}_e$$

$$\frac{\partial T}{\partial T_i} = N_i$$

$$\frac{\partial T}{\partial x} = \underset{1\times n}{\frac{\partial \lfloor N \rfloor}{\partial x}} \underset{n\times 1}{\{T\}_e} + \underset{1\times n}{\lfloor N \rfloor} \underset{n\times 1}{\frac{\partial \{T\}_e}{\partial x}} = \left\lfloor \frac{\partial N_1}{\partial x}, \frac{\partial N_2}{\partial x}, \dots, \frac{\partial N_i}{\partial x}, \dots, \frac{\partial N_n}{\partial x} \right\rfloor \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_i \\ \vdots \\ T_n \end{Bmatrix}_e$$

↑

$$\frac{\partial \{T\}_e}{\partial x} = \{0\}$$

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial N_i}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x}$$

(Jacobian matrix)

$$\frac{\partial}{\partial T_i}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial N_i}{\partial x}$$

similarly:

$$\frac{\partial T}{\partial y} = \left\lfloor \frac{\partial N_1}{\partial y}, \frac{\partial N_2}{\partial y}, \dots, \frac{\partial N_i}{\partial y}, \dots, \frac{\partial N_n}{\partial y} \right\rfloor \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_i \\ \vdots \\ T_n \end{Bmatrix}_e$$

$$\frac{\partial T}{\partial z} = \left\lfloor \frac{\partial N_1}{\partial z}, \frac{\partial N_2}{\partial z}, \dots, \frac{\partial N_i}{\partial z}, \dots, \frac{\partial N_n}{\partial z} \right\rfloor \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_i \\ \vdots \\ T_n \end{Bmatrix}_e$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} + \frac{\partial N_i}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial y} \qquad \frac{\partial N_i}{\partial z} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial z} + \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial z} + \frac{\partial N_i}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial z}$$

$$\frac{\partial}{\partial T_i}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial N_i}{\partial y} \qquad \frac{\partial}{\partial T_i}\left(\frac{\partial T}{\partial z}\right) = \frac{\partial N_i}{\partial z}$$

derivative of J_e with respect to temperature T_i:

$$\frac{\partial J_e}{\partial T_i} = \int_{\Omega_e} \lambda \left(\underset{1\times n}{\frac{\partial \lfloor N \rfloor}{\partial x}} \cdot \frac{\partial N_i}{\partial x} + \underset{1\times n}{\frac{\partial \lfloor N \rfloor}{\partial y}} \cdot \frac{\partial N_i}{\partial y} + \underset{1\times n}{\frac{\partial \lfloor N \rfloor}{\partial z}} \cdot \frac{\partial N_i}{\partial z} \right) d\Omega_e - \int_{\Omega_e} q_V N_i \, d\Omega_e$$

$$+ \int_{\Gamma_{qe}} (\alpha(\underset{1\times n}{\lfloor N \rfloor} \underset{n\times 1}{\{T\}_e} - T_\infty) N_i + q N_i) \, d\Gamma_{qe}$$

derivatives of J_e with respect to temperatures $T_1, T_2, \ldots, T_i, \ldots, T_n$:

$$\frac{\partial J_e}{\partial T_1}, \frac{\partial J_e}{\partial T_2}, \ldots, \frac{\partial J_e}{\partial T_i}, \ldots \frac{\partial J_e}{\partial T_n}$$

$\downarrow$

$$\underset{n\times 1}{\frac{\partial J_e}{\partial \{T\}_e}} = \underset{n\times n}{[h]_e} \, \underset{n\times 1}{\{T\}_e} + \underset{n\times 1}{\{F\}_e}$$

$\{F\}_e$: thermal load vector in a finite element

$[h]_e$: local conductivity matrix

components of the local conductivity matrix:

$$h_{ij} = \int_{\Omega_e} \lambda \left(\frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial z} \cdot \frac{\partial N_j}{\partial z} \right) d\Omega_e$$

components of the thermal load vector:

$$F_{ie} = -\int_{\Omega_e} q_V N_i \, d\Omega_e + \int_{\Gamma_{qe}} \alpha\big(\underset{1\times n}{\lfloor N \rfloor}\underset{n\times 1}{\{T\}_e} - T_\infty\big) N_i \, d\Gamma_{qe} + \int_{\Gamma_{qe}} q N_i \, d\Gamma_{qe}$$

heat generation — convection & radiation — applied thermal flux

3 ways of heat transfer

- conduction

$$q = -\lambda \cdot grad(T)$$

rate of heat flow per unit area (HEAT FLUX)

- convection

$$q = \alpha(T - T_{\infty})$$

film coefficient

surface temperature

bulk emperature

Medium (fluid)	Free convection	Forced convection
gas (air)	5–30	30–500
water	30–300	300–20000
oil	5–100	30–3000
liquid metals	50–500	500–20000

$$\left(\frac{W}{m^2K}\right)$$

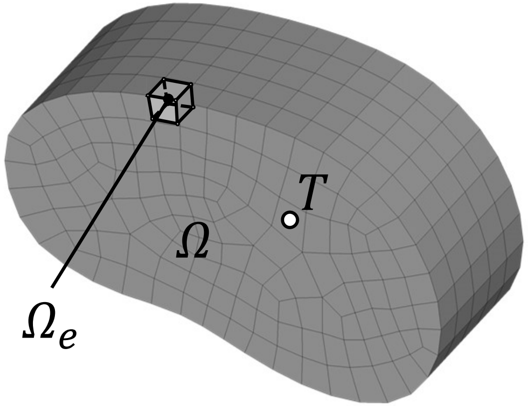
- radiation

$$\alpha = \mathrm{f}(T) = c\,\frac{T^4 - T^4}{T - T_{\infty}}$$

Minimalization of equivalent functional J for the entire FE model:

NOE – no. of FEs
NON – no. of nodes

$NON = NDOF$



To find the solution of heat transfer, the derivative of functional J with respect to the global vector of nodal temperatures is minimized:

$$\underset{NON\times 1}{\frac{\partial J}{\partial \{T\}}} = \underset{NON\times NON}{[H]} \cdot \underset{NON\times 1}{\{T\}} + \underset{NON\times 1}{\{F\}} = \underset{NON\times 1}{\{0\}}$$

global conductivity matrix ($[H]$)

global vector of nodal temperatures ($\{T\}$)

global thermal load vector ($\{F\}$)

Including boundary conditions:

$$N = NON - NOF$$

number of known nodal temperatures

Set of algebraic equation after taking boundary conditions into account:

$$\underset{N \times N}{[H]} \cdot \underset{N \times 1}{\{T\}} + \underset{N \times 1}{\{F\}} = \underset{N \times 1}{\{0\}}$$